Functions

Part Two

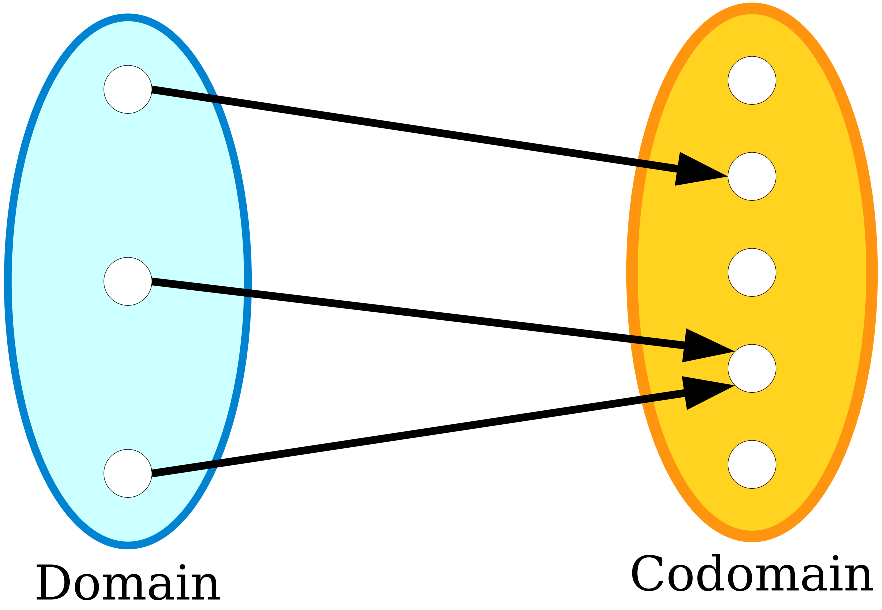
Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write $\boldsymbol{f : A \rightarrow B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.



The function must be defined for each element of its domain.

The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.

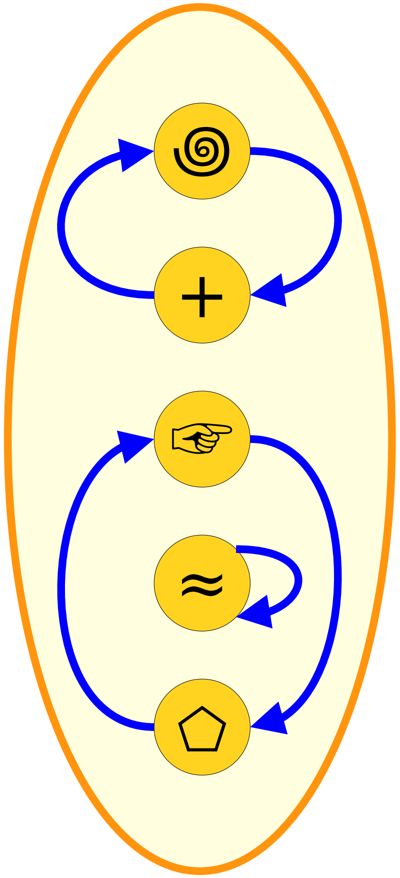
Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** when the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.



Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) when different inputs always map to different outputs.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection when this FOL statement is true:

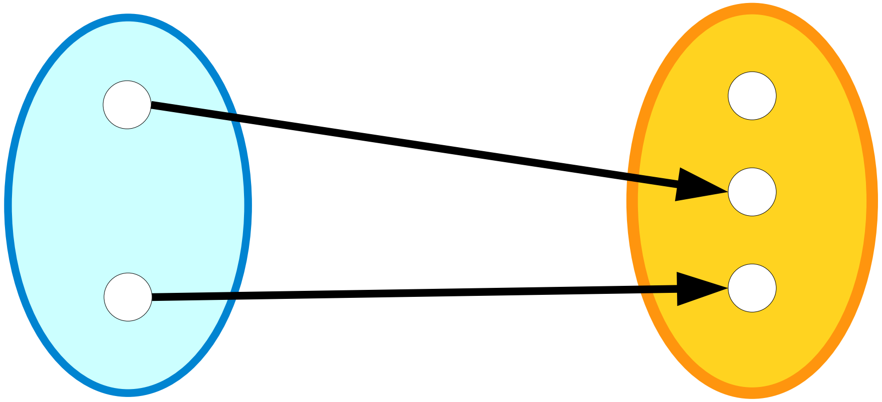
$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)



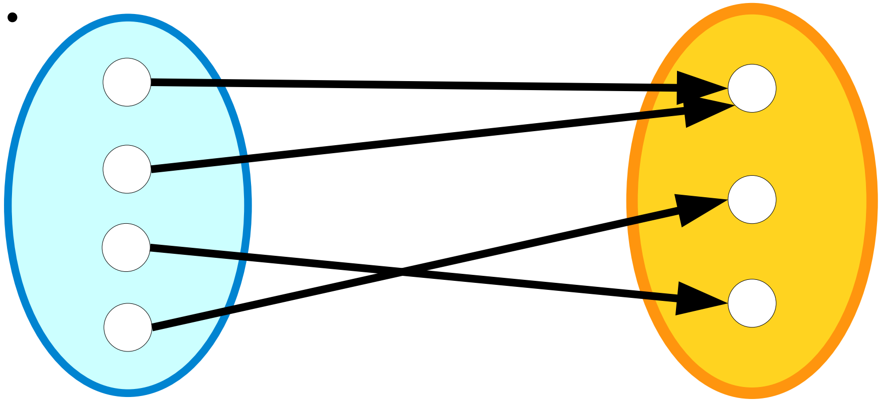
Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) when this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.



		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

New Stuff!

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

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Given the predicates

$Bird(b)$, which says b is a bird;
$Heron(h)$, which says h is a heron; and
$Feathers(x)$, which says x has feathers,

translate the theorem into first-order logic.

Answer at
https://cs103.stanford.edu/pollev

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translate the theorem into first-order logic.

$$\underbrace{(\forall b.\,(Bird(b) \rightarrow Feathers(b)))}_{\substack{\text{All birds}\\ \text{have feathers}}} \rightarrow \underbrace{(\forall h.\,(Heron(h) \rightarrow Feathers(h)))}_{\substack{\text{All herons}\\ \text{have feathers}}}$$

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Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

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Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary bird *b*. Since *b* is a bird, *b* has feathers.

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Consider an arbitrary bird b. Since b is a bird, b has feathers. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds have feathers.
  - Here, we ***proved*** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.

- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

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Proving vs. Assuming

- If we ***assume*** the statement

  $$\forall x. P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, have feathers.

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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
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$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
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$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\uparrow \atop f \text{ is an involution.}} \quad \rightarrow \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\uparrow \atop f \text{ is surjective.}}$$

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Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)} \rightarrow \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}$$

Prove this.

We've said that we need to *prove* this statement. How do we do that?

What do you do to prove
$\forall b \in A$. [something]?

Answer at
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To ***prove*** that this is true…

Have the reader pick an arbitrary x. Then prove A is true for that choice of x.

Proof Outline

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There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

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Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

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Prove this.

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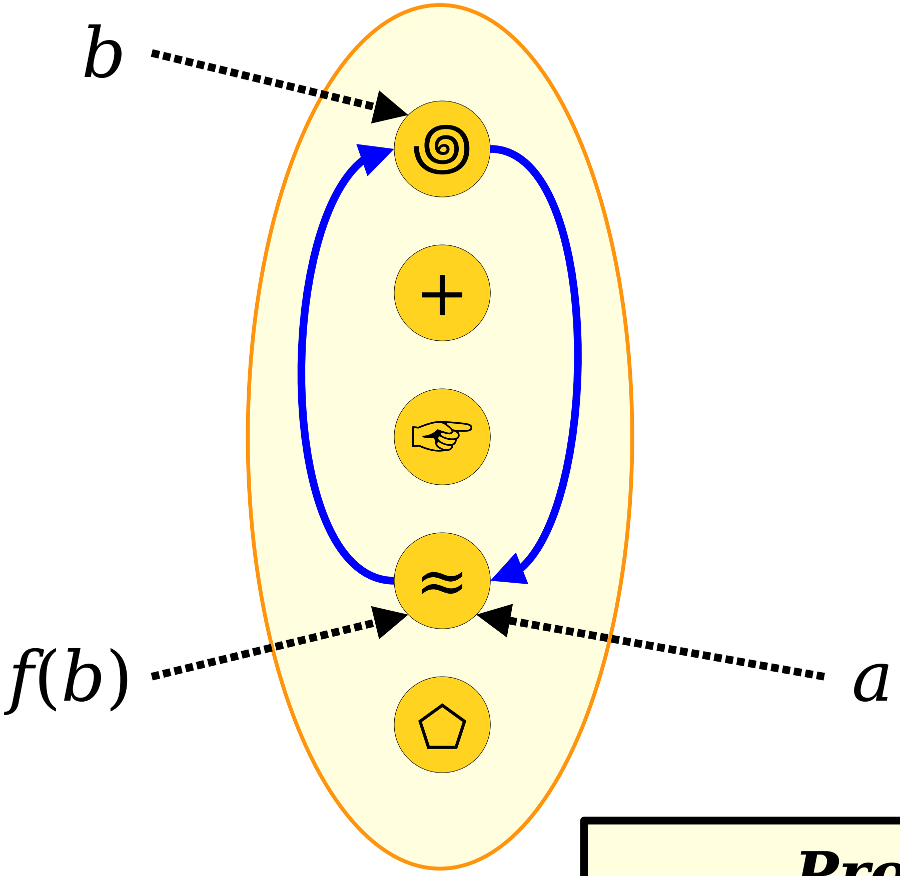
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.



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Proof: Pick any involution $f : A \to A$.

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2. Pick an arbitrary $b \in A$.
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Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$.

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Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$.

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The Two-Column Proof Organizer

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	***What We Need to Prove***
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$ $a_1 = a_2$ $)$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

What We're Assuming

$f : A \to A$ is an involution.

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$a_1 \in A$

$a_2 \in A$

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We need to prove
this implication. So
we assume the antecedent
and prove the consequent.

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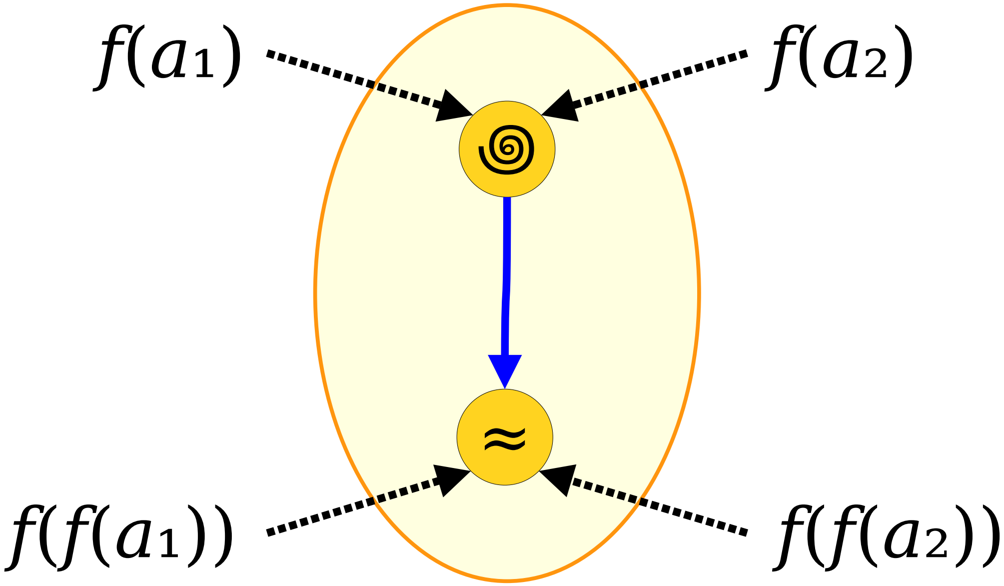

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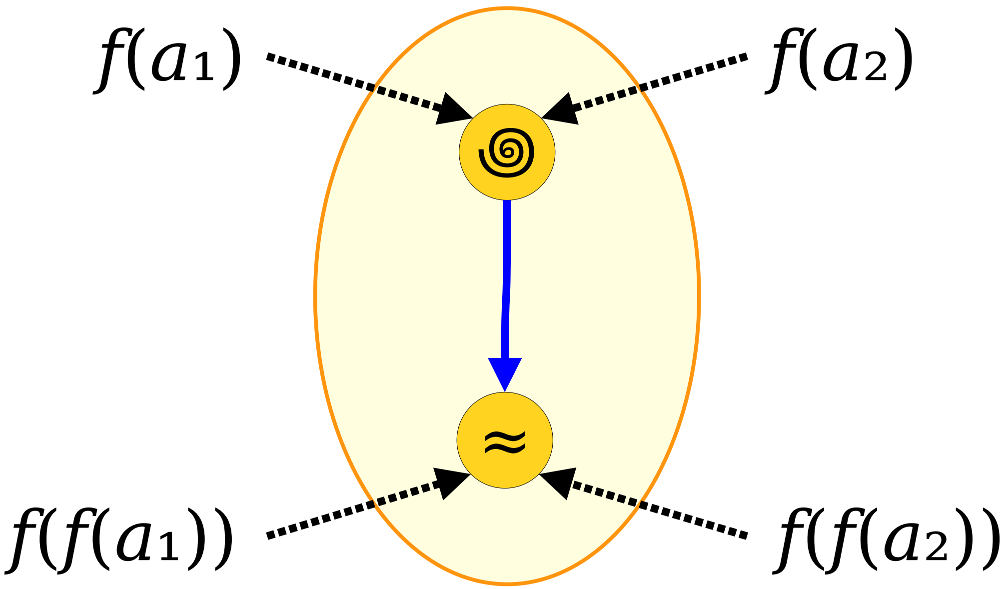

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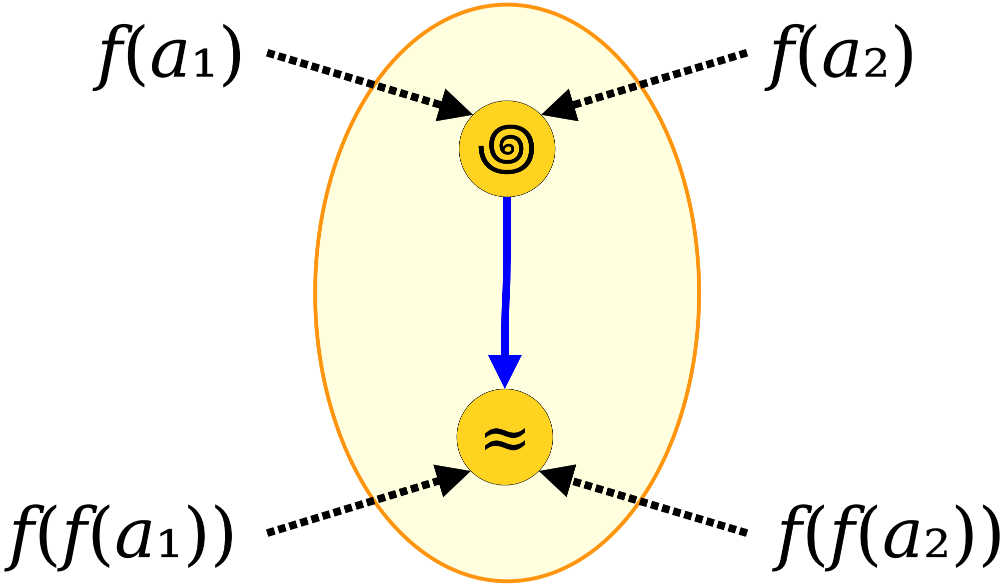

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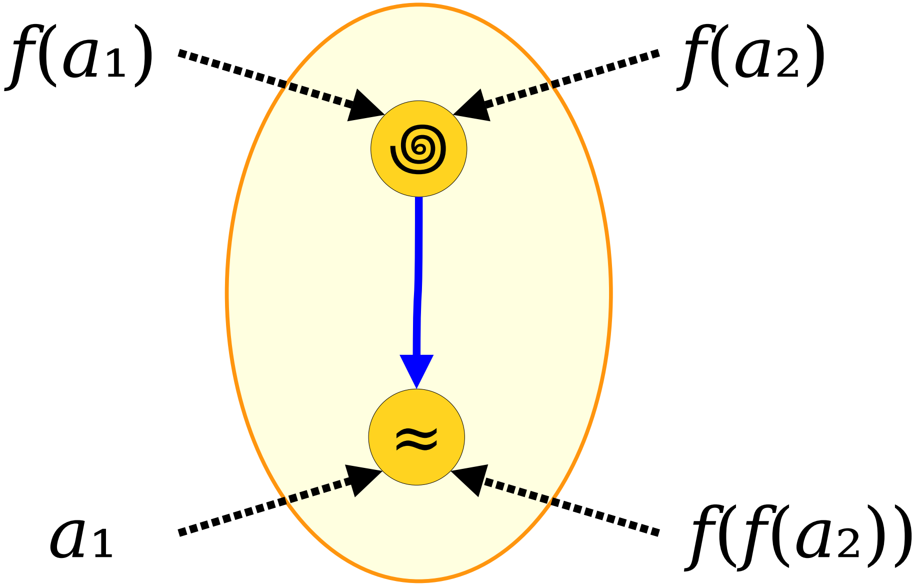

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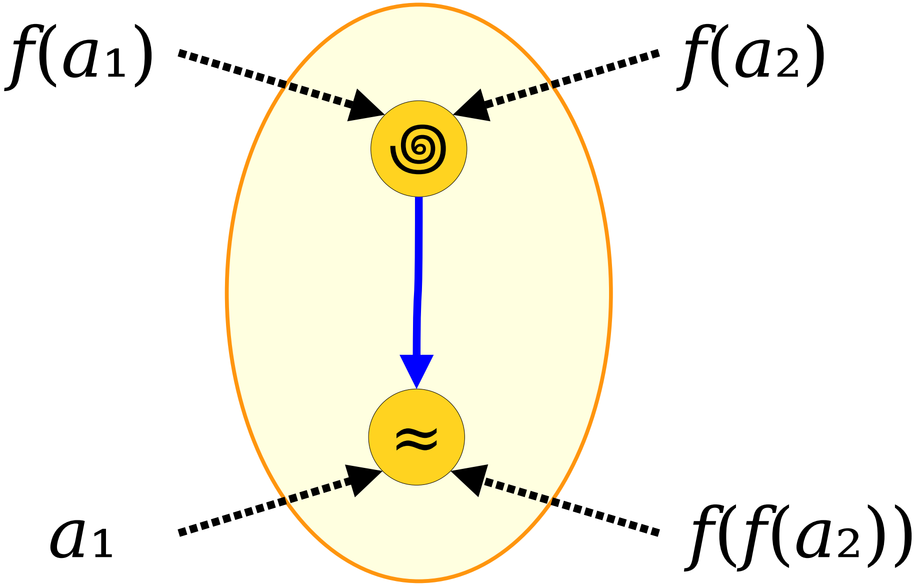

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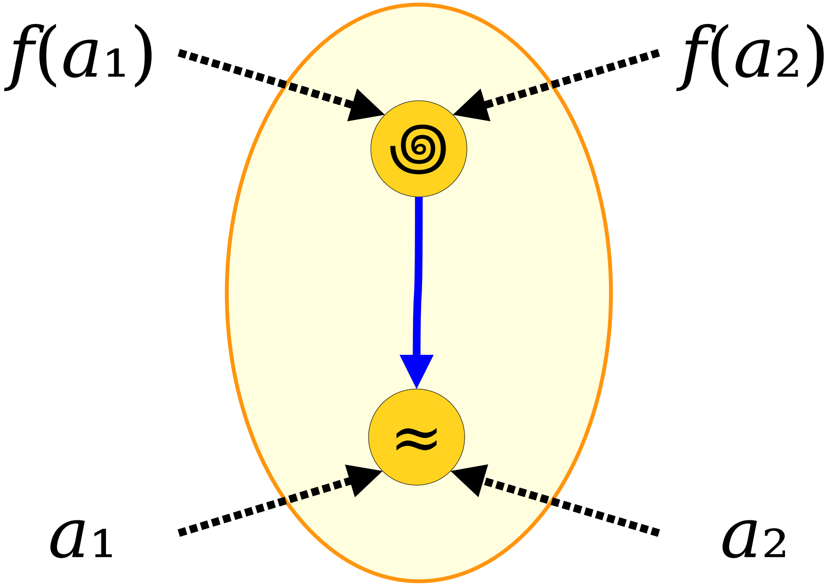

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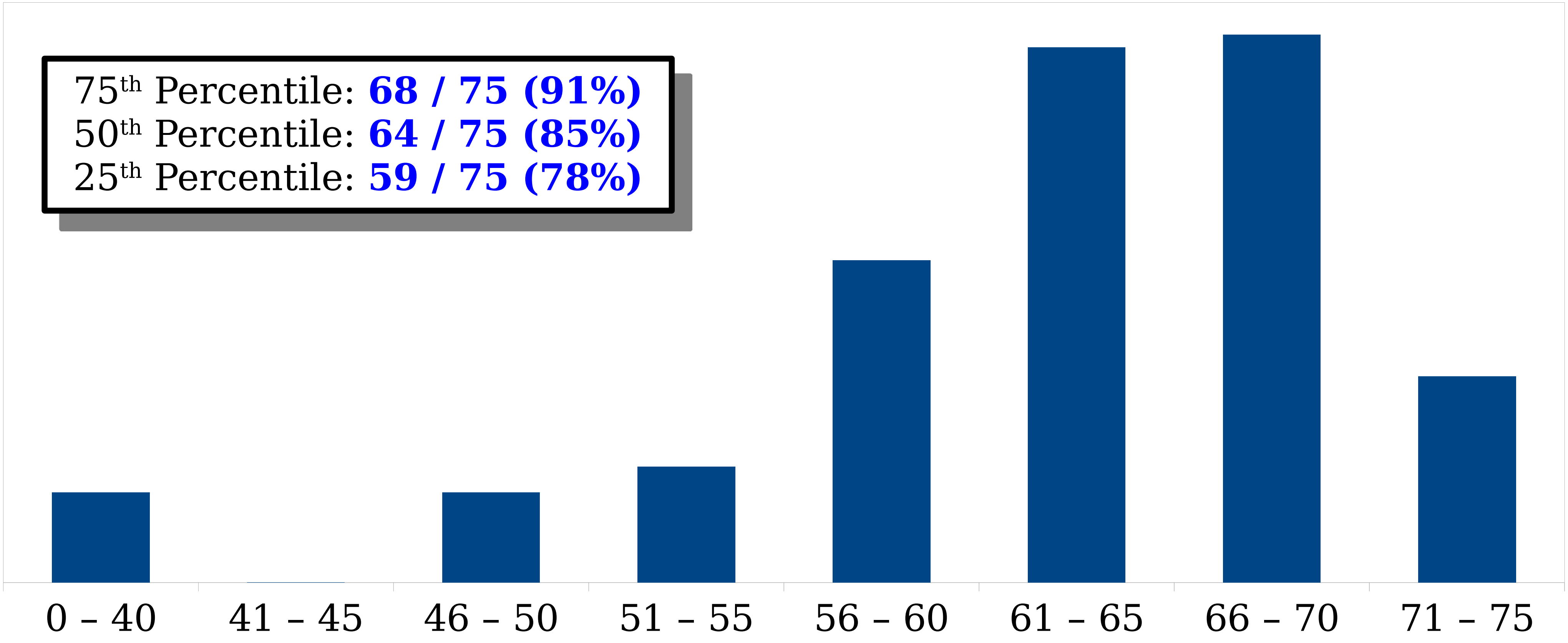
Time-Out for Announcements!

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set One Graded

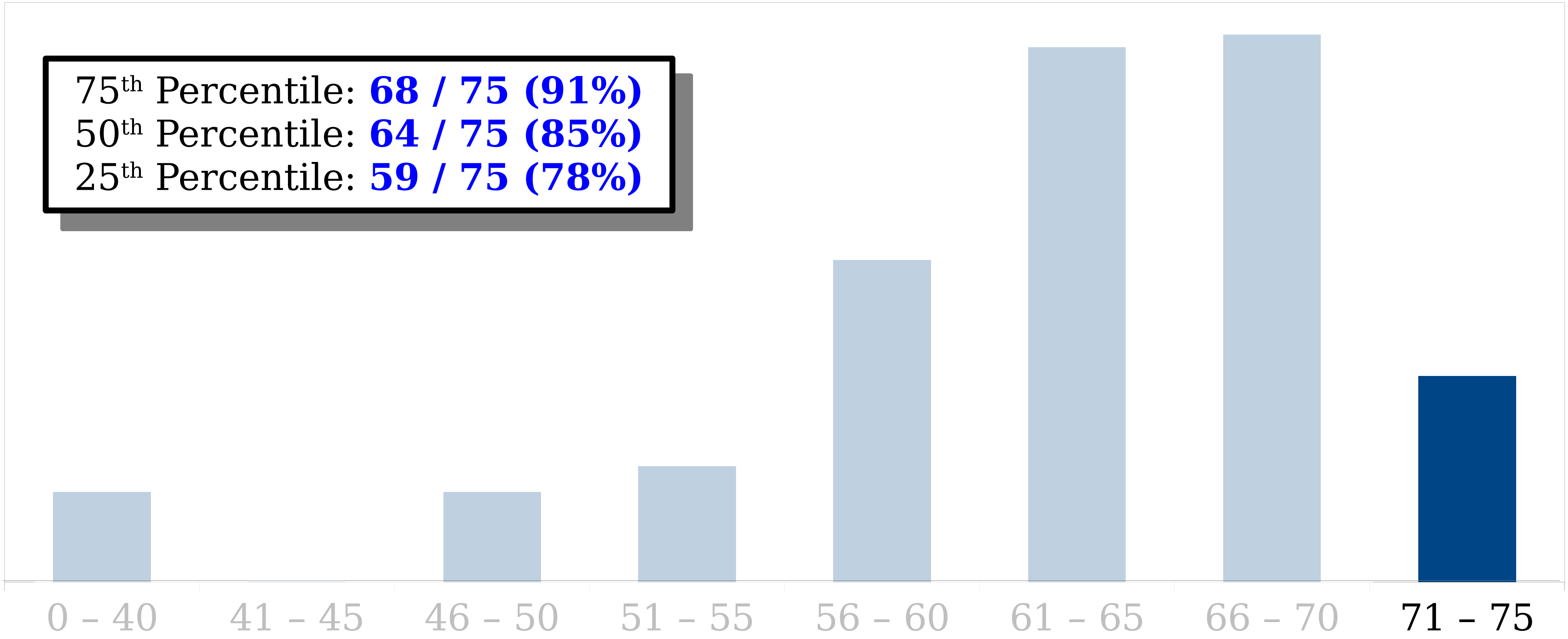


75th Percentile: **68 / 75 (91%)**
50th Percentile: **64 / 75 (85%)**
25th Percentile: **59 / 75 (78%)**

Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

Problem Set One Graded



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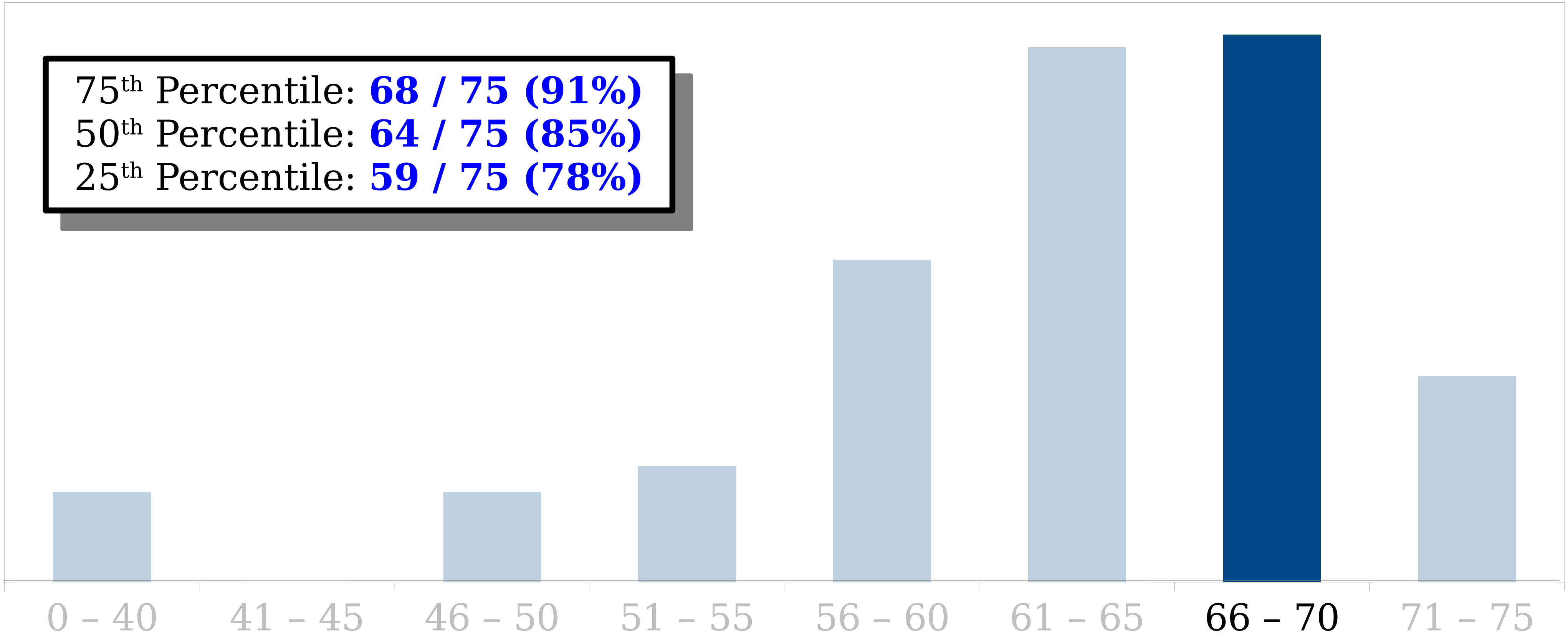
"Great job! Look over your feedback for some tips on how to tweak things for next time."

Problem Set One Graded



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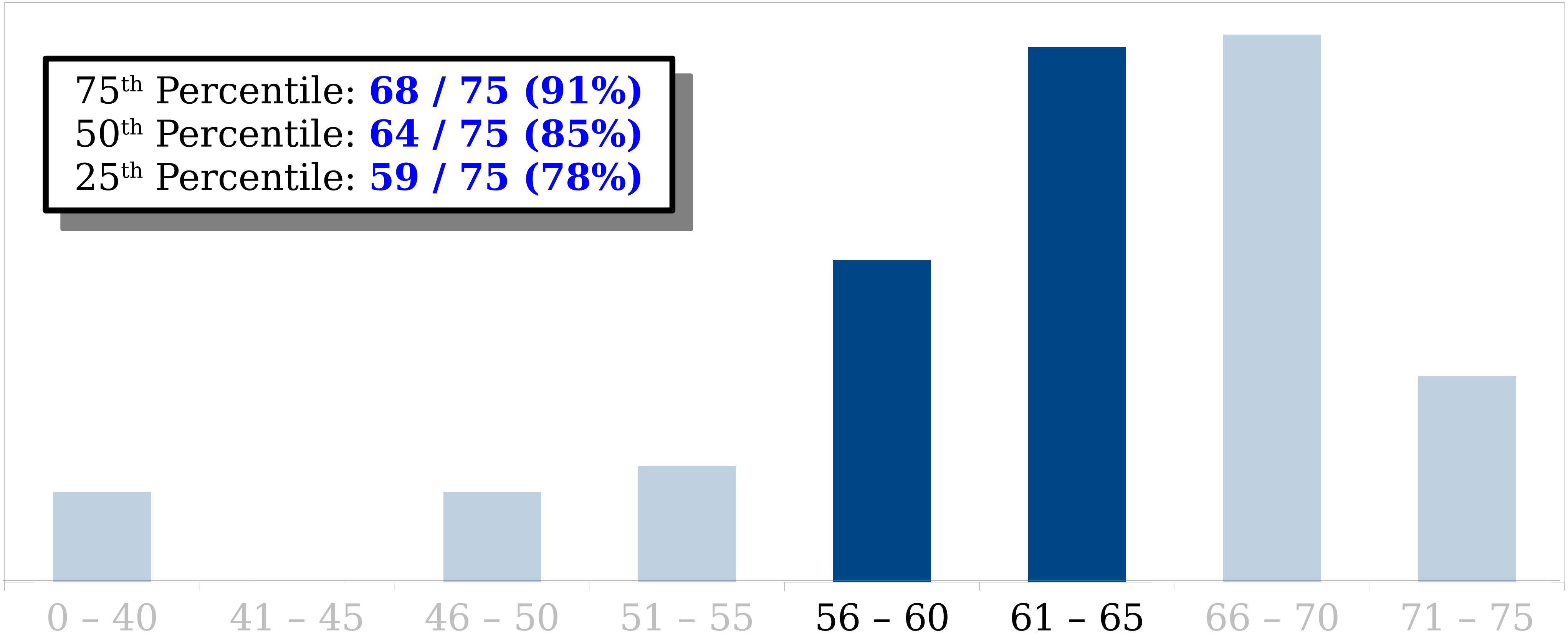
"You're almost there! Review the feedback on your submission and see what to focus on for next time."

Problem Set One Graded



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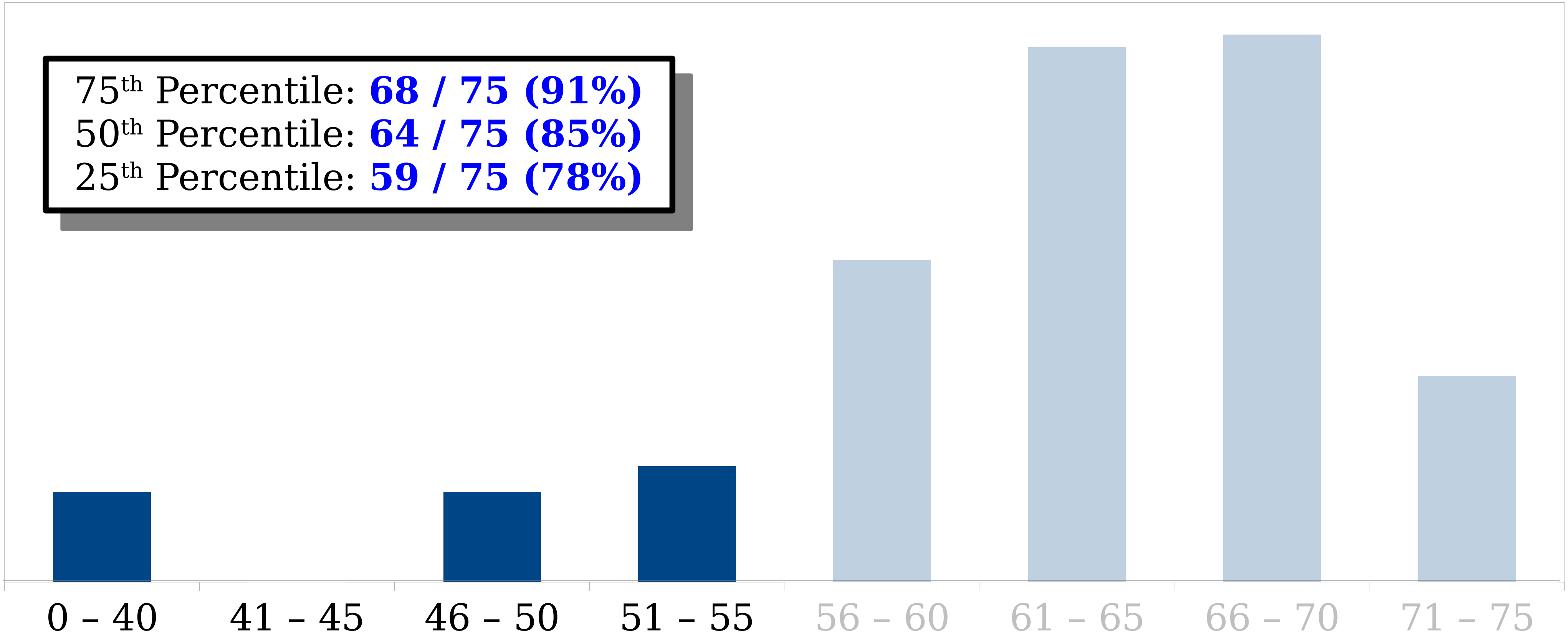
"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded



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"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"

What Not to Think

- “Well, I guess I’m just not good at math.”
  - For most of you, this is your first time doing any rigorous proof-based math.
  - Don’t judge your future performance based on a single data point.
  - Life advice: have a growth mindset!
- “Hey, I did above the median. That’s good enough.”
  - There’s always some area where you can improve. Take the time to see what that is.

Regrade Requests

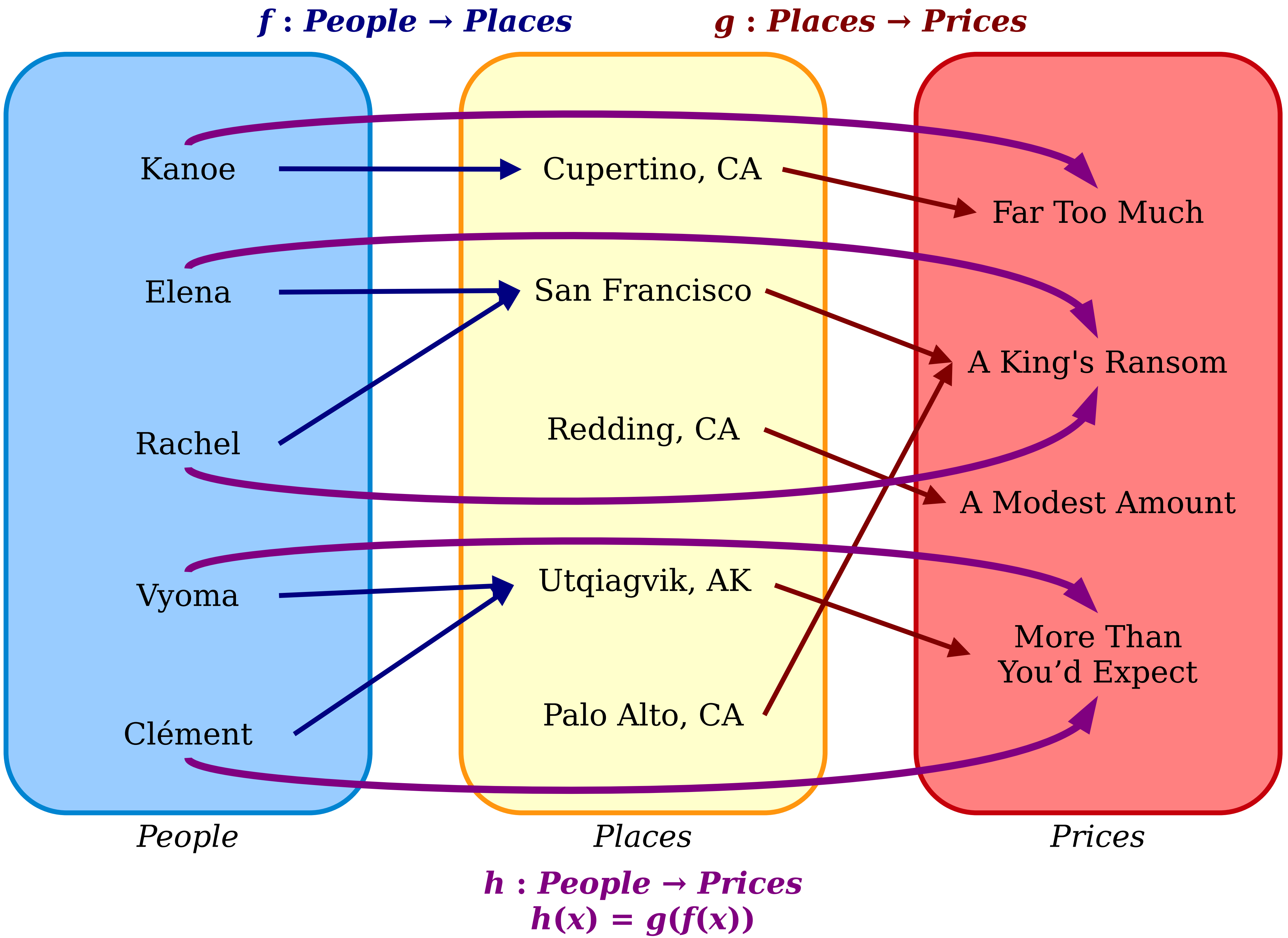
- We’re human. We make mistakes. And we’re happy to correct them!
- Regrades will open on Gradescope 48 hours after grades are released. They close one week after grades are released.
- Notes on regrades:
  - Please be civil. We make mistakes. We’re happy to correct them.
  - We have to grade what you submitted; we can’t take any clarifications into account during regrades.
  - Regrades are for where we made deductions we shouldn’t have, rather than for the magnitude of deductions.

Essential Action Items

- ***Review your feedback.***
  - Don’t just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
  - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
  - (Especially for Q8, Q10) Look at our solutions and see if there’s any neat lessons you can draw from them.
- ***Come to us with questions.***
  - Anything you’re not sure about? That’s what we’re here for! Come to office hours, ask questions on EdStem, etc.

Back to CS103!

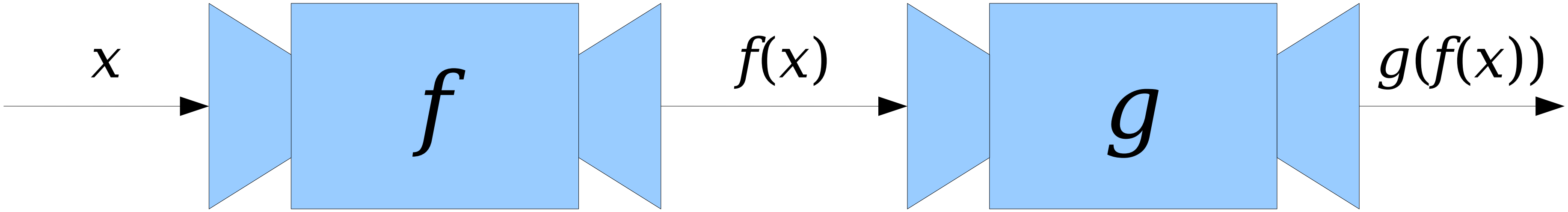
Function Composition



f : *People* → *Places*
g : *Places* → *Prices*
Kanoe
Elena
Rachel
Vyoma
Clément
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You'd Expect
People
Places
Prices
h : *People* → *Prices*
$h(x) = g(f(x))$

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

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Let's write this out separately and simplify things a bit.

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Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection. $\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$g \circ f$ is an injection. $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection. $\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	$g(f(a_1)) \neq g(f(a_2))$
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
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$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
$(g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

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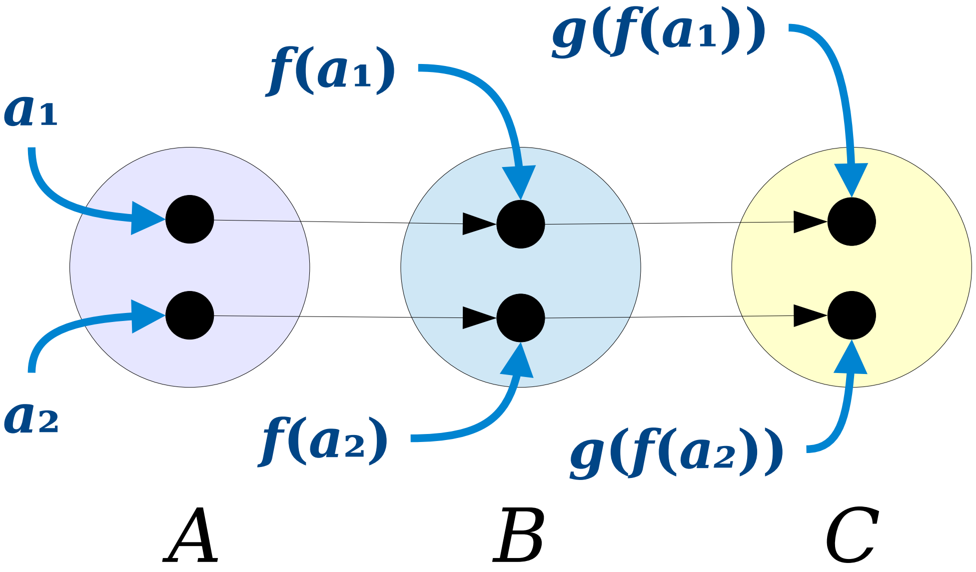
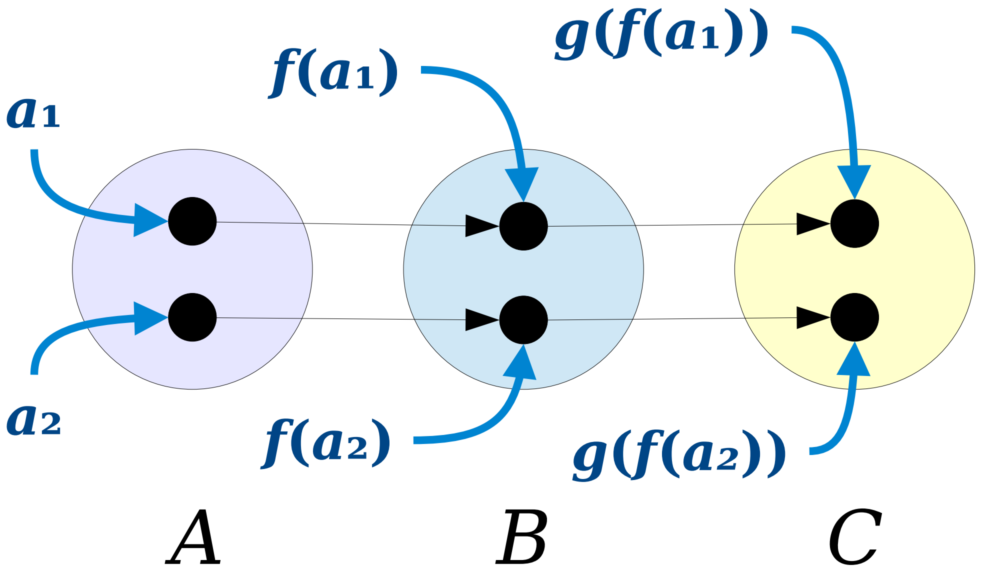


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



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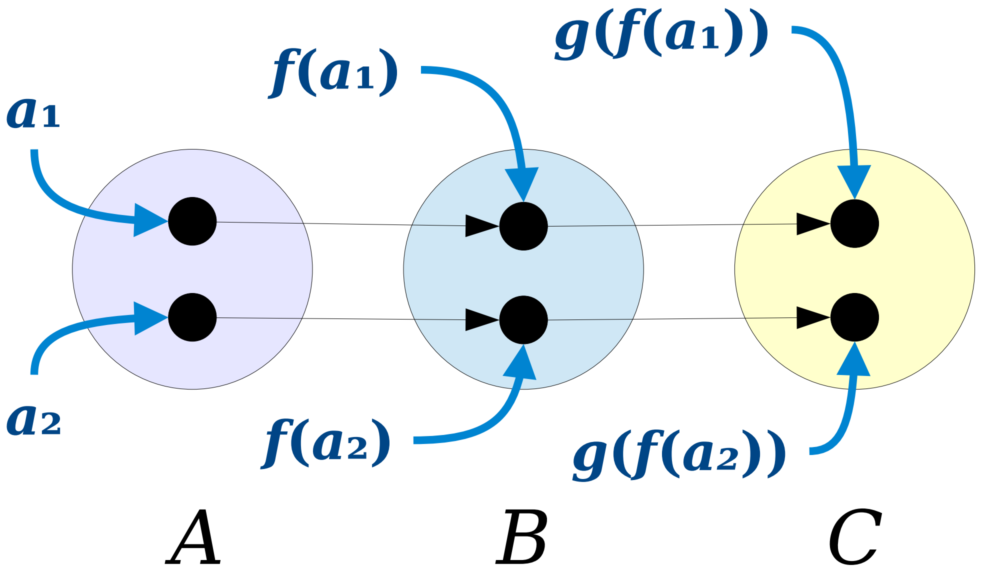
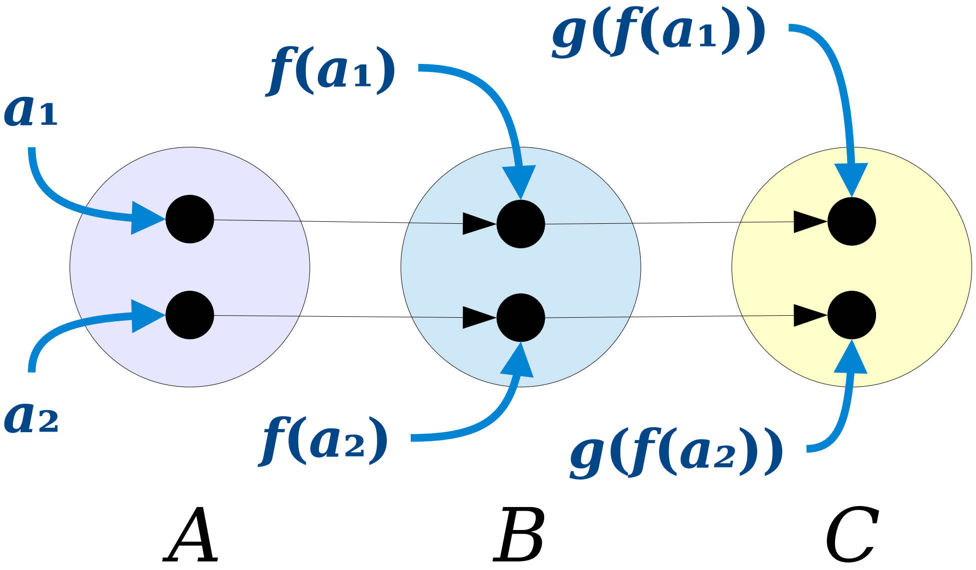


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



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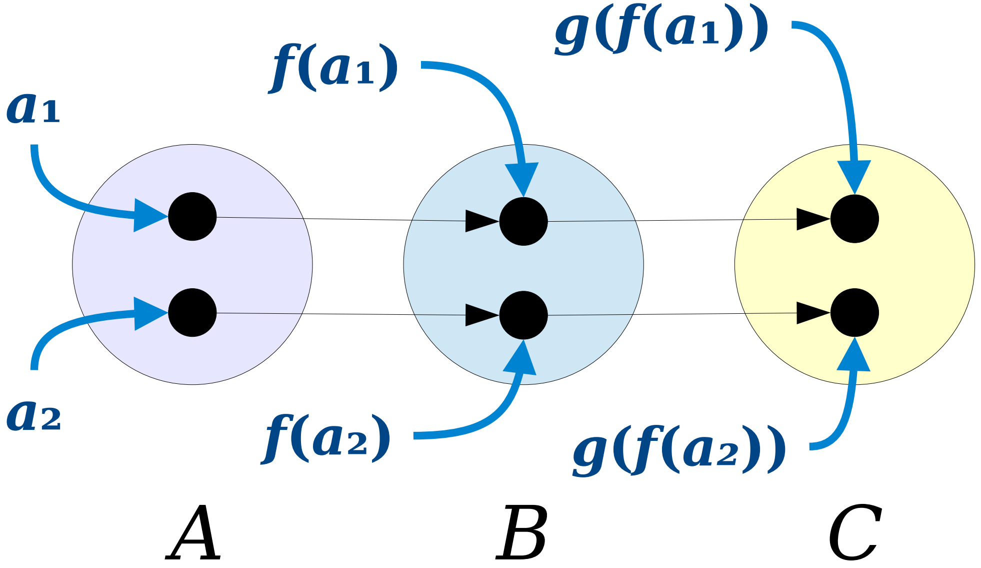
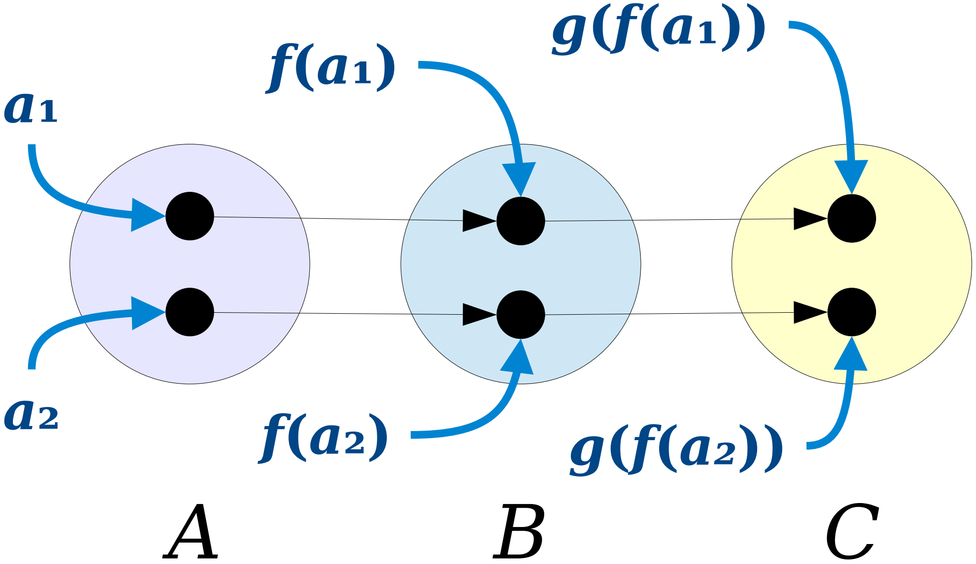
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$.



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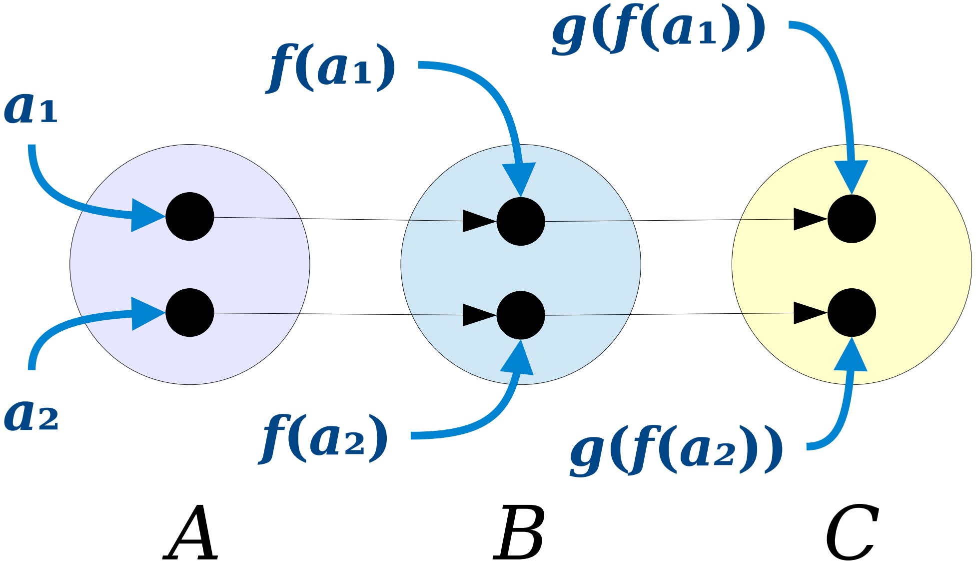

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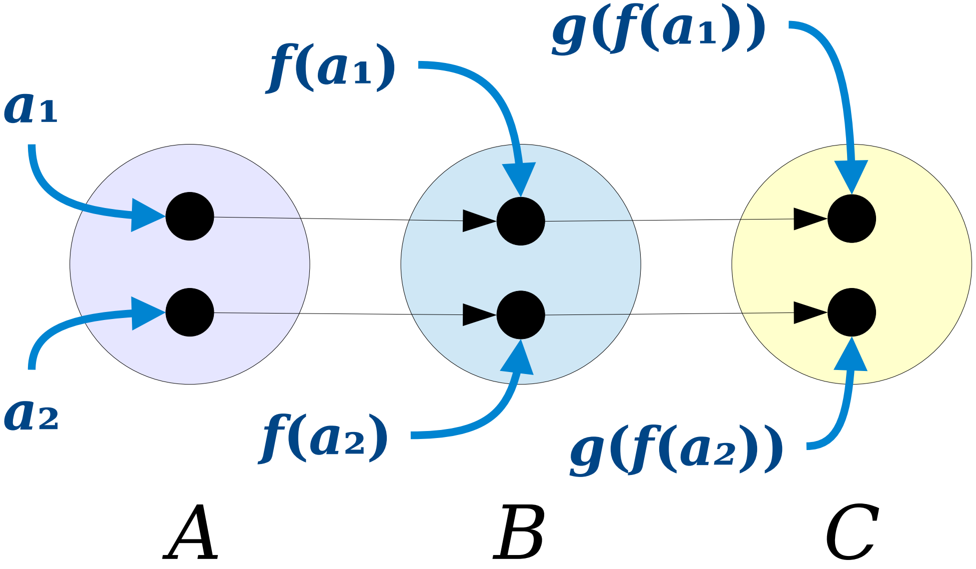

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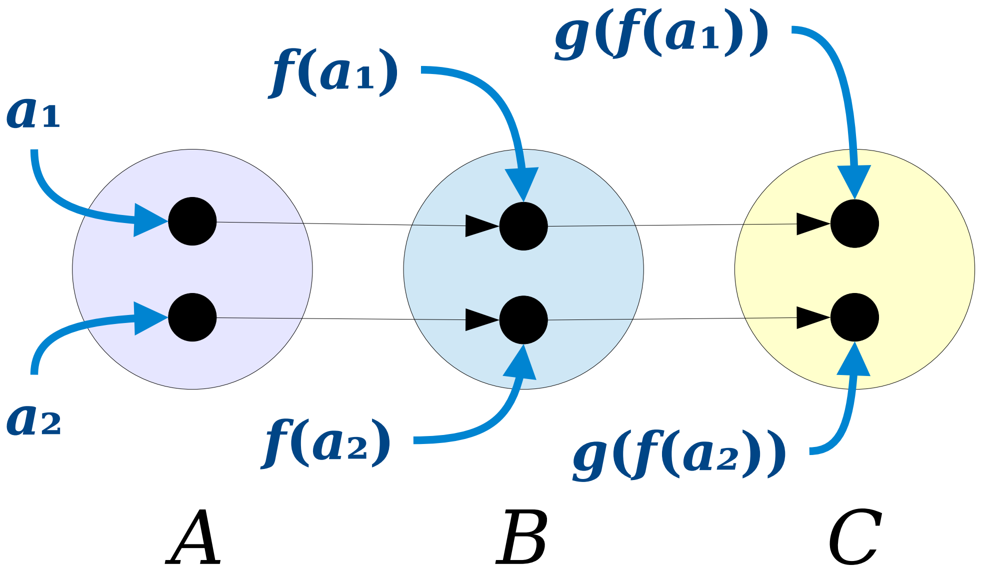
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



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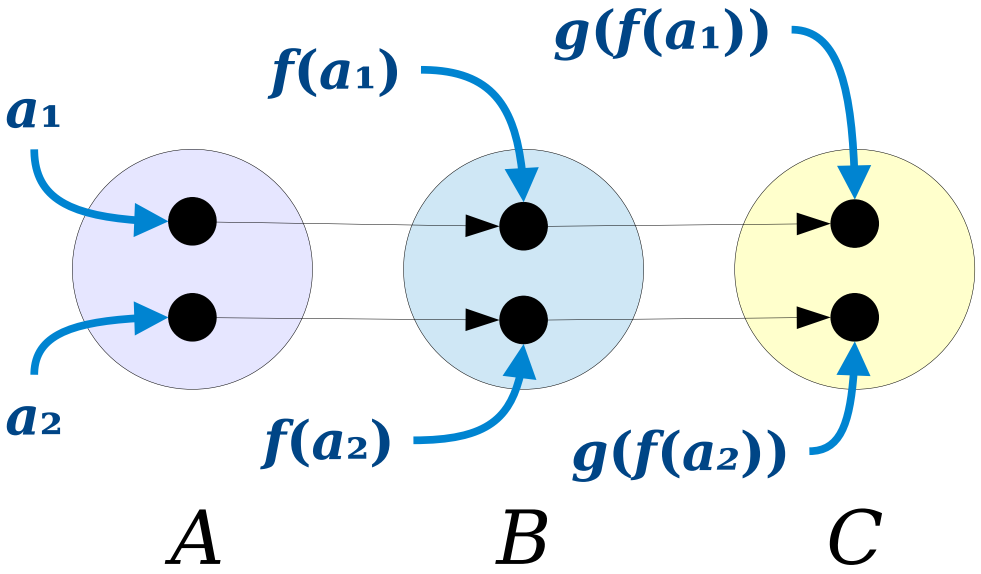
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.



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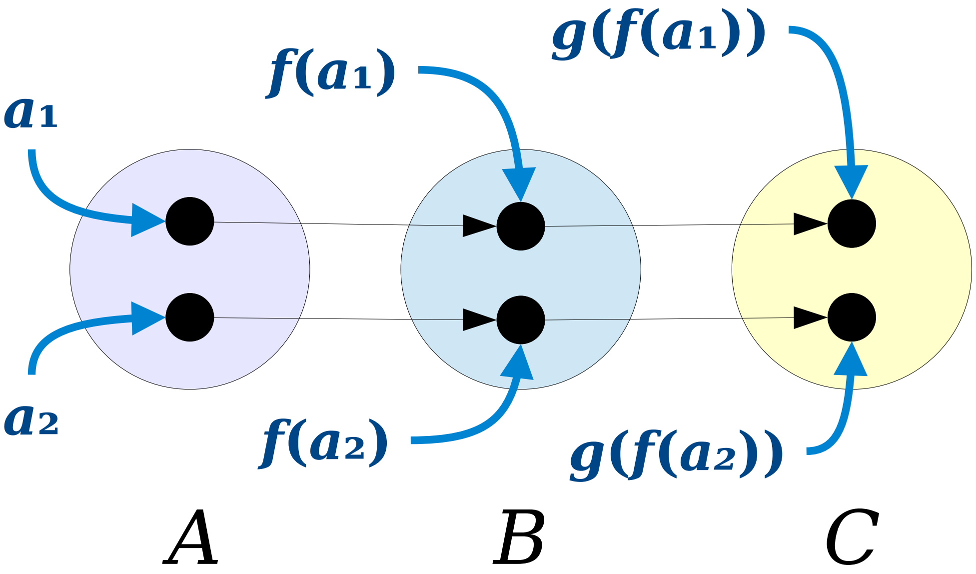
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



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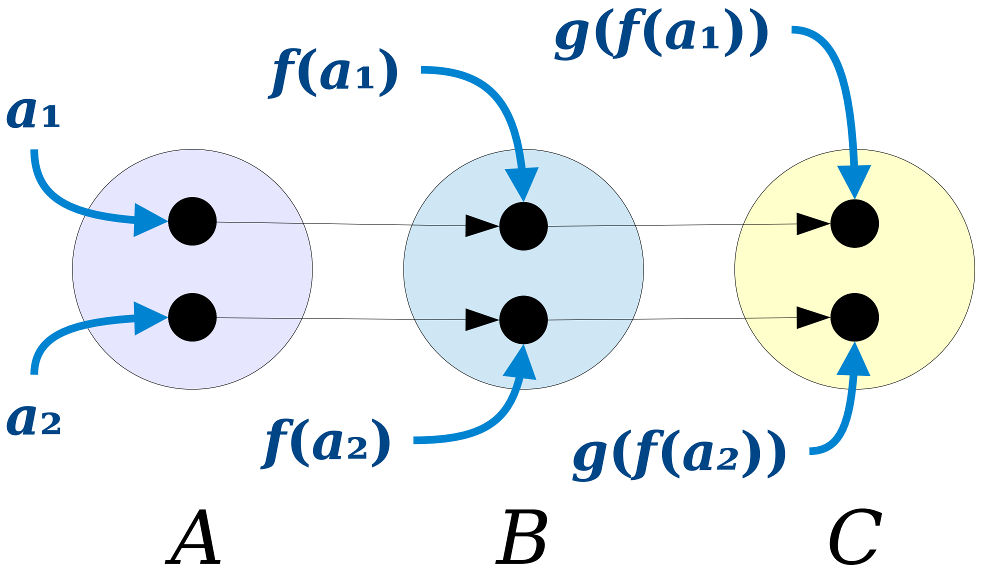
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



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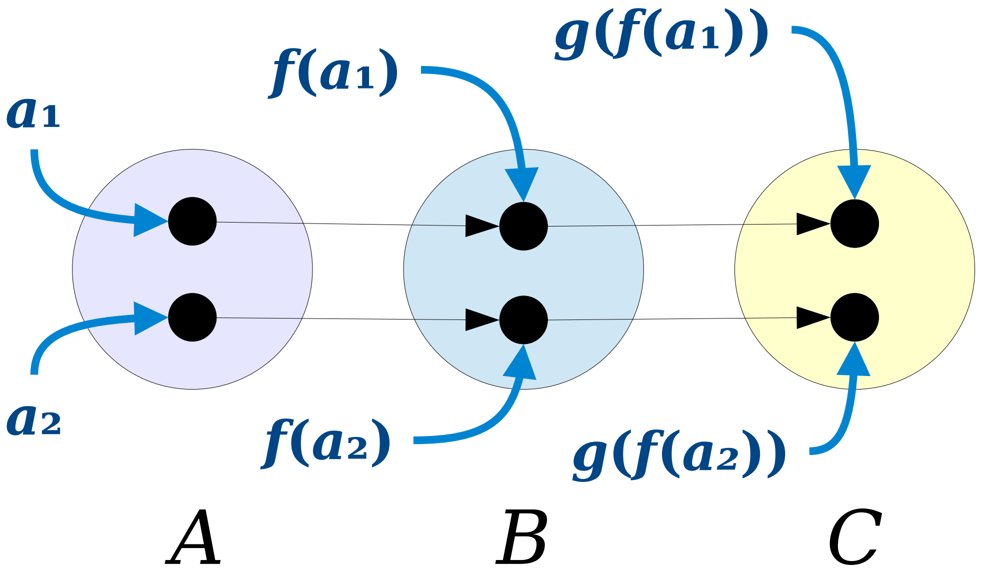
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Proofs on Sets***
  - How to rigorously establish set-theoretic results.

Appendix: Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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What does it mean for $g \circ f : A \to C$ to be surjective?

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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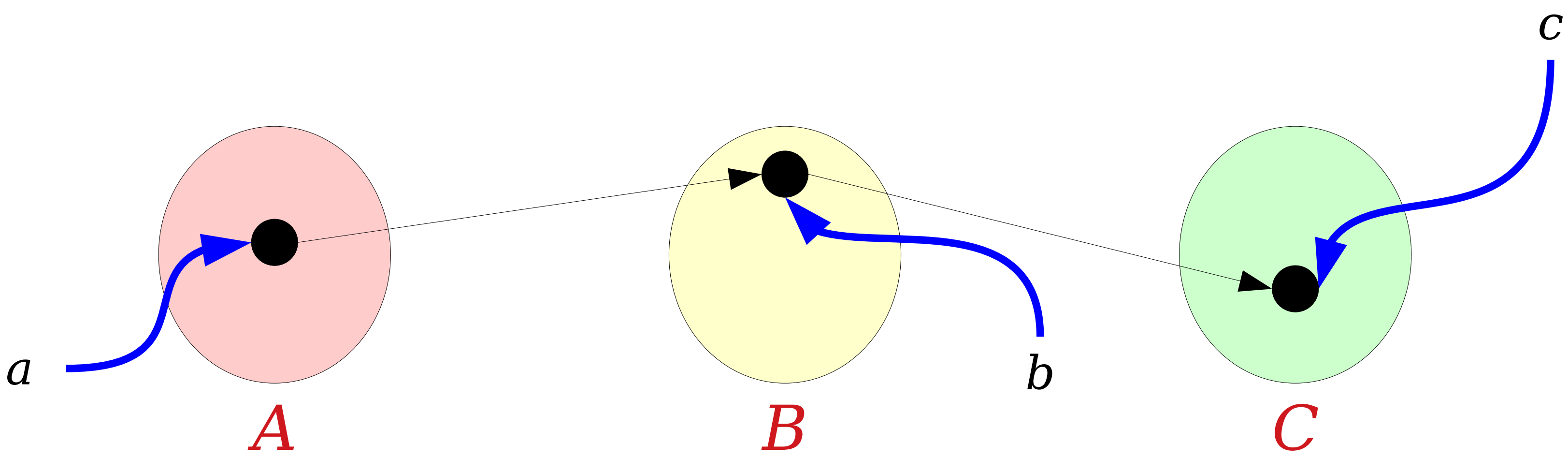

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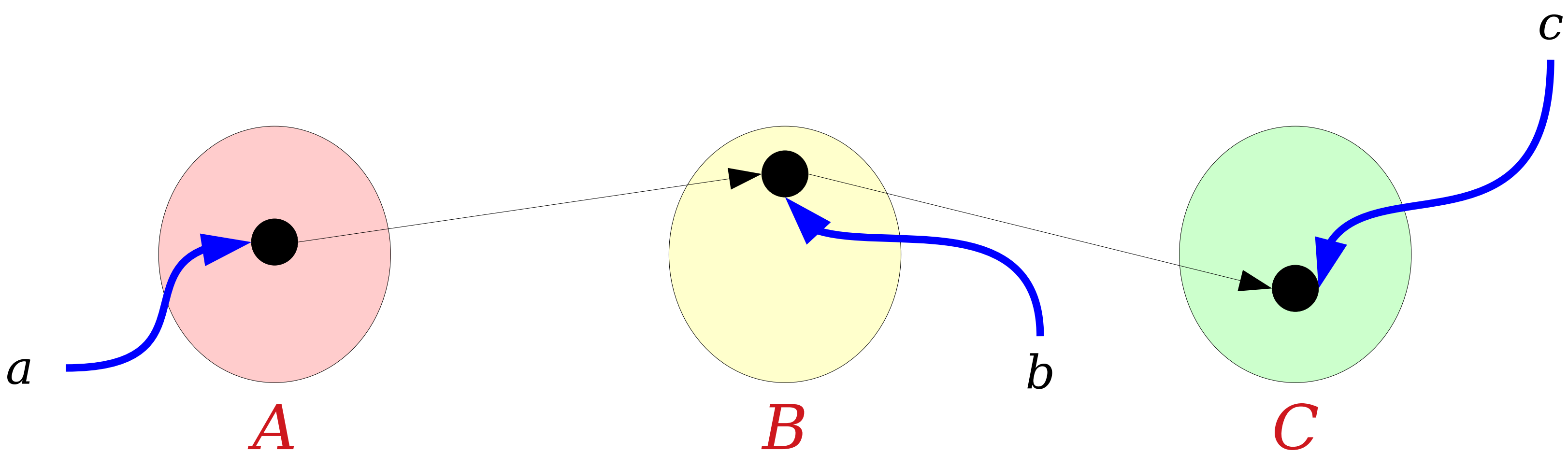
Consider any $c \in C$.



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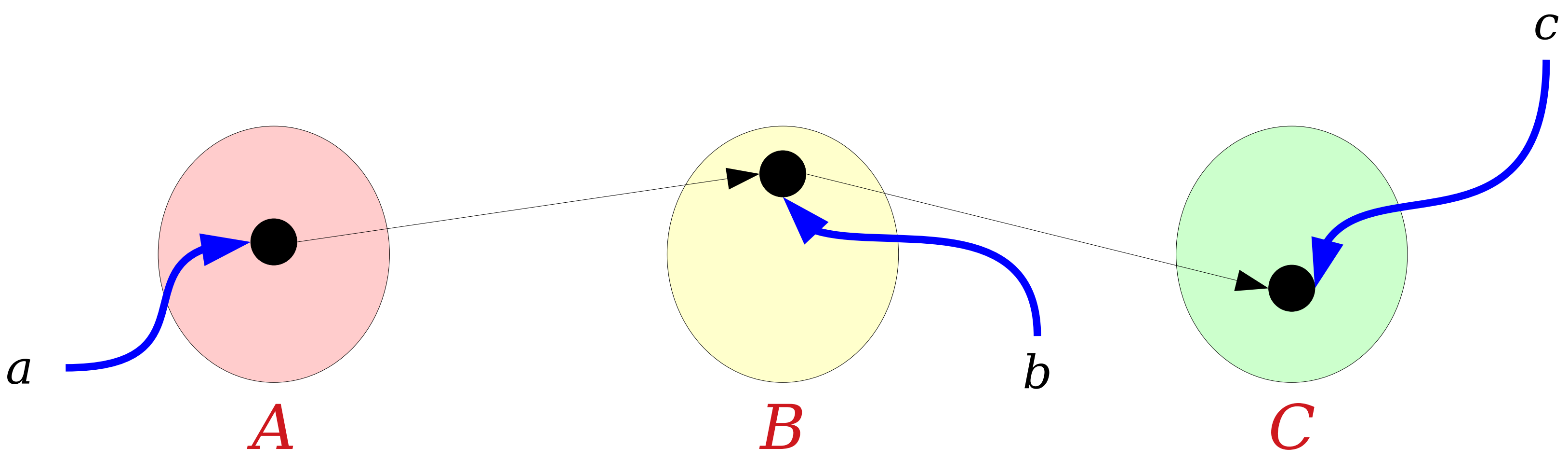
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$.



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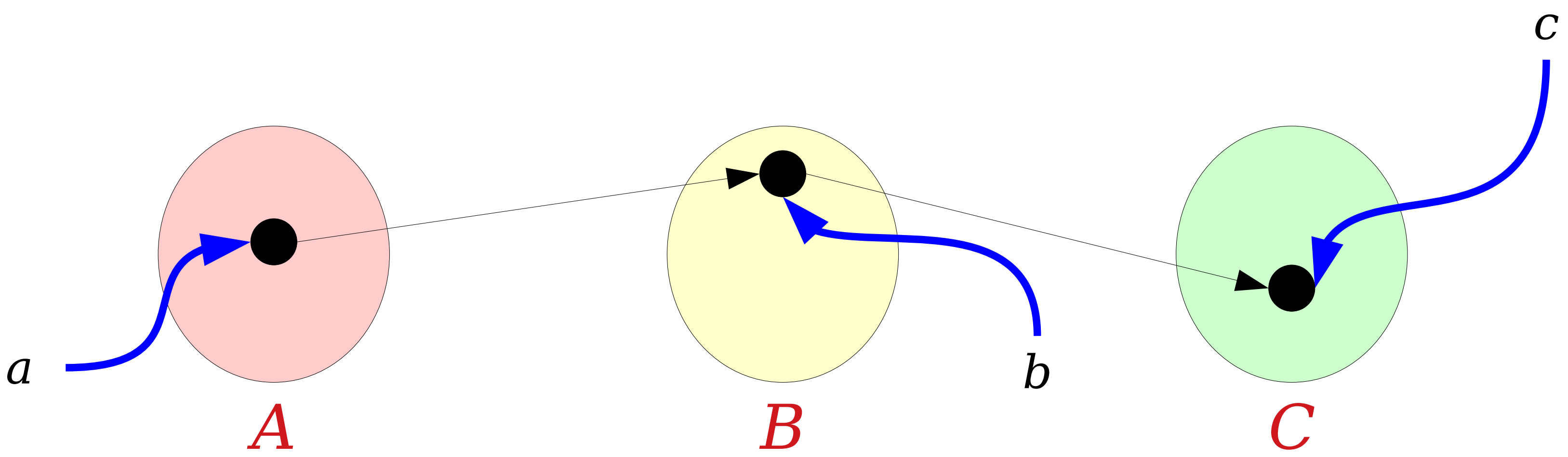
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



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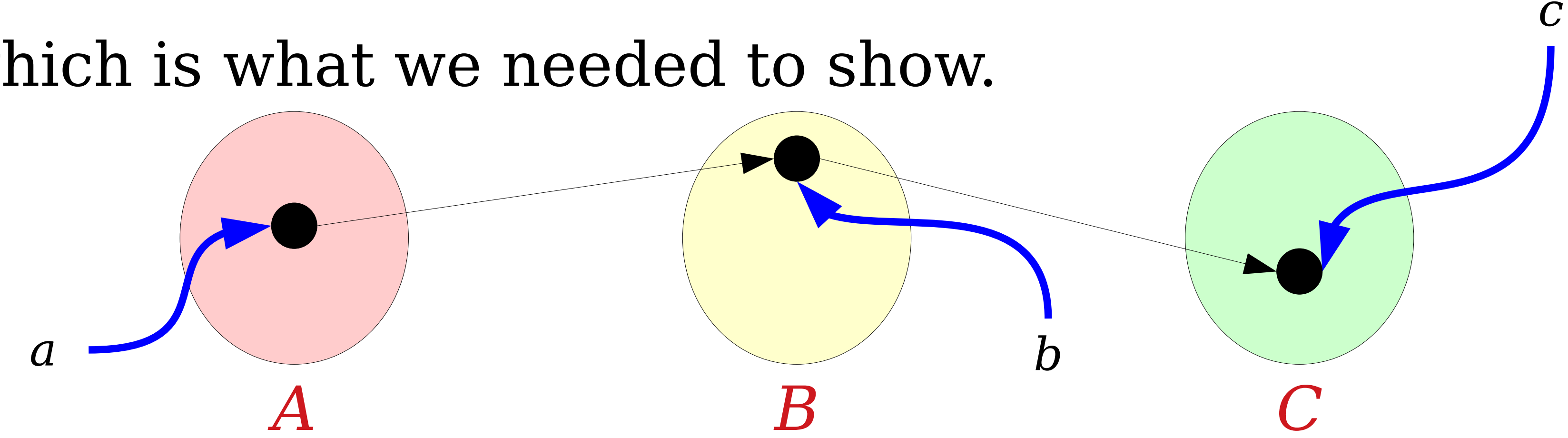
$$g(f(a)) = g(b) = c,$$

which is what we needed to show.



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